1. “On Gravitational Waves” [p. 154]

[*Einstein 1918a*]

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The important question of how gravitational fields propagate was treated by me in an academy paper one and a half years ago.[1] However, I have to return to the subject matter since my former presentation is not sufficiently transparent and, furthermore, is marred by a regrettable error in calculation. [2]

As before, I limit myself to the case where the space-time continuum that is under consideration deviates only very little from a “Galilean” one. In order to be able [3] to write for all indices

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu}, \qquad (1)$$

we select, as is customary in the special theory of relativity, the time variable as purely imaginary, i.e., we put

$$x_4 = it,$$

where t denotes the “light time.” In (1) $\delta_{\mu\nu} = 1$ or $\delta_{\mu\nu} = 0$ depending upon $\mu = \nu$ or $\mu \neq \nu$, respectively. The $\gamma_{\mu\nu}$ are small quantities compared to 1, and represent the deviation of the continuum from one that is free of fields; under Lorentz transformations, they form a tensor of rank two.

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[1] These *Sitzungsber.* (1916), pp. 688 ff. [1]

§1. Solutions of the Approximation Equations for the Gravitational Field by Means of Retarded Potentials

We start with the field equations

$$-\sum_{\alpha}\frac{\partial}{\partial x_\alpha}\begin{Bmatrix}\mu\nu\\ \alpha\end{Bmatrix}+\sum_{\alpha}\frac{\partial}{\partial x_\nu}\begin{Bmatrix}\mu\alpha\\ \alpha\end{Bmatrix}+\sum_{\alpha\beta}\begin{Bmatrix}\mu\alpha\\ \beta\end{Bmatrix}\begin{Bmatrix}\nu\beta\\ \alpha\end{Bmatrix}-\sum_{\alpha\beta}\begin{Bmatrix}\mu\nu\\ \alpha\end{Bmatrix}\begin{Bmatrix}\alpha\beta\\ \beta\end{Bmatrix}$$
$$= -\kappa\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right), \qquad (2)$$

[p. 155] which are valid for arbitrary coordinate systems.[2] $T_{\mu\nu}$ is the energy tensor of matter and T is the associated scalar $\sum_{\alpha\beta} g^{\alpha\beta}T_{\alpha\beta}$. One gets the approximation equations

[5]
$$\sum_{\alpha}\left(\frac{\partial^2\gamma_{\mu\nu}}{\partial x_\alpha^2}+\frac{\partial^2\gamma_{\alpha\alpha}}{\partial x_\mu\partial x_\nu}-\frac{\partial^2\gamma_{\mu\alpha}}{\partial x_\nu\partial x_\alpha}-\frac{\partial^2\gamma_{\nu\alpha}}{\partial x_\mu\partial x_\alpha}\right) = 2\kappa\left(T_{\mu\nu}-\frac{1}{2}\delta_{\mu\nu}\sum_{\alpha}T_{\alpha\alpha}\right) \qquad (2a)$$

if one considers all terms which are of *n*th order in $\gamma_{\mu\nu}$ as small quantities of order n and, simultaneously, limits the evaluation of both sides of equation (2) to terms of the lowest order. Multiplying this equation by $-\frac{1}{2}\delta_{\mu\nu}$ with summation over μ and ν, one next gets (by change of the indices) the scalar equation

$$\sum_{\alpha\beta}\left(-\frac{\partial^2\gamma_{\alpha\alpha}}{\partial x_\beta^2}+\frac{\partial^2\gamma_{\alpha\beta}}{\partial x_\alpha\partial x_\beta}\right) = \kappa\sum_{\alpha}T_{\alpha\alpha}$$

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[4] [2] We refrain here from an introduction of the "λ-term" (see these *Sitzungsber.* [1917], p. 142).

If this equation is multiplied by $\delta_{\mu\nu}$ and added to equation (2a), then the second term on the right-hand side of (2a) cancels out. The left-hand side can be written more comprehensively if one introduces, instead of the $\gamma_{\mu\nu}$, the functions

$$\gamma'_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\sum_{\alpha}\gamma_{\alpha\alpha}. \tag{3}$$

The equation then takes the form:

$$\sum_{\alpha}\frac{\partial^2\gamma'_{\mu\nu}}{\partial x_\alpha^2} - \sum_{\alpha}\frac{\partial^2\gamma'_{\mu\alpha}}{\partial x_\nu \partial x_\alpha} - \sum_{\alpha}\frac{\partial^2\gamma'_{\nu\alpha}}{\partial x_\mu \partial x_\alpha} + \delta_{\mu\nu}\sum_{\alpha\beta}\frac{\partial^2\gamma'_{\alpha\beta}}{\partial x_\alpha \partial x_\beta} = 2\kappa T_{\mu\nu}\,. \tag{4}$$

However, this equation can be simplified considerably if one demands the $\gamma'_{\mu\nu}$ to satisfy not only equation (4) but also the relations

$$\sum_{\alpha}\frac{\partial\gamma'_{\mu\alpha}}{\partial x_\alpha} = 0. \tag{5}$$

On first sight it seems strange that the 10 equations (4) for 10 functions $\gamma'_{\mu\nu}$ should allow for 4 additional and arbitrary conditions without becoming overdetermined. But the justification of this procedure is seen from the following. Equations (2) are covariant with respect to arbitrary substitutions, i.e., they are satisfied for an arbitrarily chosen coordinate system. Upon the introduction of a new coordinate system, the $g_{\mu\nu}$ of the new system depend upon 4 arbitrary functions which define the transformation of the coordinates. These 4 equations can now be chosen such [p. 156] that the $g_{\mu\nu}$ of the new system satisfy four arbitrarily prescribed relations. We imagine them to be such that they transform into equations (5) for the approximations we are interested in. These latter equations, therefore, represent our prescrip- [6] tion according to which the coordinate system has to be chosen. One obtains in place of (4), due to (5), the simple equations

$$\sum_{\alpha}\frac{\partial^2\gamma'_{\mu\nu}}{\partial x_\alpha^2} = 2\kappa T_{\mu\nu}\,. \tag{6}$$

It is seen from (6) that gravitational fields propagate at the speed of light. With given $T_{\mu\nu}$, the $\gamma_{\mu\nu}$ can be calculated from them in the manner of retarded potentials. If x, y, z, t are the real-valued coordinates of the point $x_1, x_2, x_3, \frac{x_4}{i}$ under consideration, i.e., the point for which the $\gamma'_{\mu\nu}$ are to be calculated, and $x_\circ, y_\circ, z_\circ$ are the spatial coordinates of the volume element $dV_\circ$, r the spatial distance between the latter and the point under consideration, then one gets

$$\gamma'_{\mu\nu} = -\frac{\kappa}{2\pi}\int\frac{T_{\mu\nu}(x_0, y_0, z_0, t-r)}{r}dV_\circ. \tag{7}$$

§2. The Energy Components of the Gravitational Field

Some time ago[3] I gave the energy components of the gravitational field explicitly in case the choice of coordinates satisfies the condition

$$g = |g_{\mu\nu}| = 1,$$

a condition which, for the approximation considered here, is equivalent to

$$\gamma = \sum_\alpha \gamma_{\alpha\alpha} = 0.$$

This, however, is in general not satisfied with our present choice of coordinates. Therefore, the most simple method for obtaining the energy components follows a separate consideration.

However, we have to consider the following difficulty. Our field equations (6) are correct only up to the first order of magnitude, while the energy equations—as is easily concluded—are small of the second order of magnitude. But we reach our
[8] goal easily by the following consideration. The energy components $\mathfrak{T}_\mu^\sigma$ (of matter)

[7] [3]*Ann. d. Phys.* 49 (1916), eq. (50).

and t_μ^σ (of the gravitational field) satisfy, according to the general theory, the relations [p. 157]

$$\sum_\sigma \frac{\partial \mathfrak{T}_\mu^\sigma}{\partial x_\sigma} + \frac{1}{2}\sum_{\rho\sigma} \frac{\partial g^{\rho\sigma}}{\partial x_\mu} \mathfrak{T}_{\rho\sigma} = 0,$$ [9]

$$\sum_\sigma \frac{\partial(\mathfrak{T}_\mu^\sigma + t_\mu^\sigma)}{\partial x_\sigma} = 0.$$

From these follows

$$\sum_\sigma \frac{\partial t_\mu^\sigma}{\partial x_\sigma} = \frac{1}{2}\sum_{\rho\sigma} \frac{\partial g^{\rho\sigma}}{\partial x_\mu} \mathfrak{T}_{\rho\sigma}.$$

We find the t_μ^σ if we bring the right-hand side in the form of the left-hand side, thereby using the $\mathfrak{T}_{\rho\sigma}$ from the field equations. In case of the approximation considered here, the two factors on the right-hand side of this equation are small quantities of the first order. In order to get the t_μ^σ accurately as quantities of the second order, one only needs to substitute the two factors on the right accurately in terms of quantities of the first order. One, therefore, can replace [10] [11][1]

$$\frac{\partial g^{\rho\sigma}}{\partial x_\mu} \text{ by } -\frac{\partial \gamma_{\rho\sigma}}{\partial x_\mu}$$

$$\text{and } \mathfrak{T}_{\rho\sigma} \text{ by } T_{\rho\sigma}.$$

For t_μ^σ we introduce the $t_{\rho\sigma}$ which, under the desired approximation, deviate in value from the t_ρ^σ only in sign. With respect to the character of their indices, the $t_{\rho\sigma}$ are quantities analogous to the $T_{\mu\sigma}$. We have to determine the $t_{\mu\sigma}$ from the equation

$$\sum_\sigma \frac{\partial t_{\mu\sigma}}{\partial x_\sigma} = \frac{1}{2}\sum_{\rho\sigma} \frac{\partial \gamma_{\rho\sigma}}{\partial x_\mu} T_{\rho\sigma}. \tag{8}$$

We transform the right-hand side by observing that, due to (3),

$$\gamma_{\mu\nu} = \gamma'_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\sum_{\alpha}\gamma'_{\alpha\alpha} = \gamma'_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\gamma' \tag{3a}$$

has to be used and, furthermore, by expressing the $T_{\rho\sigma}$, with the help of (6) by means of $\gamma'_{\rho\sigma}$. One gets, after a simple rearrangement,[4]

[13][2]

$$\sum_{\sigma}\frac{\partial t_{\mu\sigma}}{\partial x_{\sigma}} = \sum_{\sigma}\frac{\partial}{\partial x_{\sigma}}\left[\frac{1}{4\kappa}\left(\sum_{\alpha\beta}\left(\frac{\partial\gamma'_{\alpha\beta}}{\partial x_{\mu}}\frac{\partial\gamma'_{\alpha\beta}}{\partial x_{\sigma}}\right) - \frac{1}{2}\frac{\partial\gamma'}{\partial x_{\mu}}\frac{\partial\gamma'}{\partial x_{\sigma}}\right)\right.$$
$$\left. - \frac{1}{8\kappa}\delta_{\mu\sigma}\left(\sum_{\alpha\beta\lambda}\left(\frac{\partial\gamma'_{\alpha\beta}}{\partial x_{\lambda}}\right)^2 - \frac{1}{2}\sum_{\lambda}\left(\frac{\partial\gamma'}{\partial x_{\lambda}}\right)^2\right)\right].$$

[p. 158] From this it follows that we satisfy the energy theorem if we set

$$4\kappa t_{\mu\sigma} = \left(\sum_{\alpha\beta}\left(\frac{\partial\gamma'_{\alpha\beta}}{\partial x_{\mu}}\frac{\partial\gamma'_{\alpha\beta}}{\partial x_{\sigma}}\right) - \frac{1}{2}\frac{\partial\gamma'}{\partial x_{\mu}}\frac{\partial\gamma'}{\partial x_{\sigma}}\right)$$
$$-\frac{1}{2}\delta_{\mu\sigma}\left(\sum_{\alpha\beta\lambda}\left(\frac{\partial\gamma'_{\alpha\beta}}{\partial x_{\lambda}}\right)^2 - \frac{1}{2}\sum_{\lambda}\left(\frac{\partial\gamma'}{\partial x_{\lambda}}\right)^2\right). \tag{9}$$

The easiest way to grasp the physical meaning of the $t_{\mu\sigma}$ results from the following consideration. The $t_{\mu\sigma}$ are for the gravitational field what the $T_{\mu\sigma}$ are for matter. But for incoherent, ponderable matter one has, under limitation to quantities of first order,

$$T_{\mu\sigma} = T^{\mu\sigma} = \rho\frac{dx_{\mu}}{ds}\frac{dx_{\sigma}}{ds}\text{, where }\left(ds^2 = -\sum_{\nu}dx_{\nu}^{\ 2}\right), \tag{10}$$

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[12] [4] The error in my previous paper [mentioned at the beginning] was that I had used $\frac{\partial\gamma'_{\rho\sigma}}{\partial x_{\mu}}$ on the right-hand side of (8) instead of $\frac{\partial\gamma_{\rho\sigma}}{\partial x_{\mu}}$. This error also necessitates a rewriting of §2 and §3 of said paper.

and where ρ is the scalar of the density of matter. The $T_{11}, T_{12}, \ldots T_{33}$, therefore, represent pressure components; T_{14}, T_{24}, T_{34} or T_{41}, T_{42}, T_{43} resp. is the vector of momentum density multiplied by $\sqrt{-1}$, or the density of the energy current, whereas T_{44} is the negative value of the energy density. The interpretation of the $t_{\mu\sigma}$ referring to the gravitational field follows from this in analogy.

As an example we next treat the field of a mass point M at rest. From (7) and (10) follows immediately

$$\gamma'_{44} = \frac{\kappa}{2\pi}\frac{M}{r} \qquad (11)$$ [14]

while all other $\gamma'_{\mu\nu}$ vanish. According to (11), (3a) and (1), one gets for the $g_{\mu\nu}$ the values that were first given by De Sitter: [15]

$$\left.\begin{matrix} -1-\frac{\kappa}{4\pi}\frac{M}{r} & 0 & 0 & 0 \\ 0 & -1-\frac{\kappa}{4\pi}\frac{M}{r} & 0 & 0 \\ 0 & 0 & -1-\frac{\kappa}{4\pi}\frac{M}{r} & 0 \\ 0 & 0 & 0 & -1+\frac{\kappa}{4\pi}\frac{M}{r} \end{matrix}\right\} \cdot \quad (11a)$$

The speed of light c which is generally given by the equation

$$0 = ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx_\mu dx_\nu$$

follows here from the relation

$$\left(1+\frac{\kappa}{4\pi}\frac{M}{r}\right)(dx^2+dy^2+dz^2) - \left(1-\frac{\kappa}{4\pi}\frac{M}{r}\right)dt^2 = 0.$$

Therefore, under the choice of coordinates we favored, the velocity of light [p. 159]

$$c = \sqrt{\frac{dx^2+dy^2+dz^2}{dt^2}} = 1 - \frac{\kappa M}{4\pi r} \qquad (12)$$

[16] depends upon the location but not upon direction. It also follows from (11a) that small, rigid bodies remain *similar* under changes of their position, whereby their linear extension, measured in coordinates, changes with $\left(1-\frac{\kappa M}{8\pi r}\right)$.

In our case, equation (9) gives for the $t_{\mu\sigma}$

[17]
$$\left.\begin{aligned} t_{\mu\sigma} &= \frac{\kappa M^2}{32\pi^2}\left(\frac{x_\mu x_\sigma}{r^6}-\frac{1}{2}\delta_{\mu\sigma}\frac{1}{r^4}\right) \text{ (for indices 1 to 3)} \\ t_{14} &= t_{24} = t_{34} = 0 \\ t_{44} &= -\frac{\kappa M^2}{64\pi^2}\cdot\frac{1}{r^4} \end{aligned}\right\} \qquad (13)$$

The values of the $t_{\mu\sigma}$ definitely depend upon the choice of coordinates, a fact
[18] Herr G. Nordström already pointed out to me in a letter some time ago.[5] If the choice of coordinates is made with the condition $|g| = 1$, for which I previously
[20] gave the $g_{\mu\nu}$ in case of a mass point with the expressions

[3]
$$\begin{aligned} g_{\mu\sigma} &= -\delta_{\mu\sigma} - \frac{\kappa M}{4\pi}\,\frac{x_\mu x_\sigma}{r^3} \qquad \text{(for indices 1 to 3)} \\ g_{14} &= g_{24} = g_{34} = 0 \\ g_{44} &= 1-\frac{\kappa M}{4\pi}\cdot\frac{1}{r}, \end{aligned}$$

then all energy components of the gravitational field vanish when one calculates them accurately to second-order quantities by means of the formula

$$\kappa t_\sigma^\alpha = \frac{1}{2}\delta_\sigma^\alpha \sum_{\mu\nu\lambda\beta} g^{\mu\nu}\begin{Bmatrix}\mu\lambda\\ \beta\end{Bmatrix}\begin{Bmatrix}\nu\beta\\ \lambda\end{Bmatrix} - \sum_{\mu\nu\lambda} g^{\mu\nu}\begin{Bmatrix}\mu\lambda\\ \alpha\end{Bmatrix}\begin{Bmatrix}\nu\sigma\\ \lambda\end{Bmatrix}.$$

One might suspect that a suitable choice of the system of reference would perhaps always get all the energy components of the gravitational field to vanish—

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[19] [5] See also E. Schrödinger, *Phys. Zeitschr.* 1 (1918), p. 4.

which would be quite remarkable. But it can easily be shown that this is not generally true. [21]

§3. The Plane Gravitational Wave

In order to find plane gravitational waves, we start from the ansatz

$$\gamma'_{\mu\nu} = \alpha_{\mu\nu} f(x_1 + ix_4), \qquad (14)$$

which satisfies the field equations (6). The $\alpha_{\mu\nu}$ are here real-valued constants and [p. 160]
f is a real-valued function of $(x_1 + ix_4)$. The equations (5) produce the relations

$$\left.\begin{aligned} \alpha_{11} + i\alpha_{14} &= 0 \\ \alpha_{21} + i\alpha_{24} &= 0 \\ \alpha_{31} + i\alpha_{34} &= 0 \\ \alpha_{41} + i\alpha_{44} &= 0 \end{aligned}\right\}. \qquad (15)$$

When the conditions (15) are met, (14) represents a possible gravitational wave. We calculate the density of its energy current $\frac{t_{41}}{i}$ in order to get a better understanding of its physical nature. Putting into equation (9) the $\gamma'_{\mu\nu}$, which are given in (15), [4]
one gets [22]

$$\frac{t_{41}}{i} = \frac{1}{4\kappa} f'^2 \left[\left(\frac{\alpha_{22} - \alpha_{33}}{2}\right)^2 + \alpha_{23}{}^2\right]. \qquad (16)$$ [23]

This result seems strange insofar as of six arbitrary constants, which occur in (14) if (15) is used, only *two* remain in (16). A wave for which $\alpha_{22} - \alpha_{33}$ and α_{23} vanish does not transport energy. This phenomenon can be deduced from the fact that such wave, in a certain sense, does not have any real existence at all, as can be derived in the simplest way from the following consideration.

We note first that with respect to (15) the scheme of the coefficients $\alpha_{\mu\nu}$ of an energy-free wave is

$$(\alpha_{\mu\nu}=) \quad \left.\begin{matrix} \alpha & \beta & \gamma & i\alpha \\ \beta & \delta & 0 & i\beta \\ \gamma & 0 & \delta & i\gamma \\ i\alpha & i\beta & i\gamma & -\alpha \end{matrix}\right\}, \tag{17}$$

where α, β, γ, δ are four mutually independent selectable numbers

Next, we look at a field-free space whose line element ds with respect to suitably chosen coordinates (x_1', x_2', x_3', x_4') can be expressed in the form

$$-ds^2 = dx_1'^2 + dx_2'^2 + dx_3'^2 + dx_4'^2. \tag{18}$$

We now introduce new coordinates x_1, x_2, x_3, x_4 by means of the substitution

$$x'_\nu = x_\nu - \lambda_\nu \phi(x_1 + ix_4). \tag{19}$$

The four λ_ν are real-valued, infinitesimally small constants, and ϕ is a real-valued function of the argument $(x_1 + ix_4)$. If quantities of the second degree in λ_ν are neglected, it follows from (18) and (19) that

$$ds^2 = -\sum_\nu dx'^2_\nu = -\sum_\nu dx_\nu^2 + 2\phi'(dx_1 + idx_4)\sum_\nu \lambda_\nu dx_\nu .$$

[p. 161] From this follow for the associated $\gamma_{\mu\nu}$ the values

$$\left(\frac{1}{\phi'}\gamma_{\mu\nu}=\right) \quad \begin{matrix} 2\lambda_1 & \lambda_2 & \lambda_3 & i\lambda_1 + \lambda_4 \\ \lambda_2 & 0 & 0 & i\lambda_2 \\ \lambda_3 & 0 & 0 & i\lambda_3 \\ i\lambda_1 + \lambda_4 & i\lambda_2 & i\lambda_3 & 2i\lambda_4 \end{matrix}$$

and from these for the $\gamma'_{\mu\nu}$

$$\left(\frac{1}{\phi'}\gamma'_{\mu\nu}=\right) \quad \left.\begin{array}{llll} \lambda_1 - i\lambda_4 & \lambda_2 & \lambda_3 & i\lambda_1 + \lambda_4 \\ \lambda_2 & -\lambda_1 - i\lambda_4 & 0 & i\lambda_2 \\ \lambda_3 & 0 & -\lambda_1 - i\lambda_4 & i\lambda_3 \\ i\lambda_1 + \lambda_4 & i\lambda_2 & i\lambda_3 & -\lambda_1 + i\lambda_4 \end{array}\right\} . \qquad (20)$$

[24] [5]

If we furthermore fix the connection between the function ϕ in (19) and the function f in (14) by the relation

$$\phi' = f, \qquad (21)$$

then it is seen that, aside from the naming of the constants, the $\gamma'_{\mu\nu}$ of (20) and the $\gamma'_{\mu\nu}$ of (14) and (17) completely agree with each other.

Those gravitational waves which transport no energy can, therefore, be generated from a field-free system by a mere coordinate transformation; their existence is (in this sense) only an *apparent one*. Real in the sense proper are, therefore, only [25] those waves, traveling along the x-axis, whose propagation corresponds to the quantities $\frac{(\gamma'_{22} - \gamma'_{33})}{2}$ and γ'_{23} (or to the quantities $\frac{(\gamma_{22} - \gamma_{33})}{2}$ and γ_{23} resp.). The two types are of the same nature and differ only in their mutual orientation. The wave field acts in deforming angles in a plane which is perpendicular to the direction of propagation. Density of the energy current, momentum, and energy are given by (16).

§4. The Emission of Gravitational Waves by Mechanical Systems

We consider an isolated mechanical system whose center of gravity shall permanently coincide with the coordinate origin. Changes within this system shall occur so slowly and its spatial extension shall be so small that the light-time corresponding to the distance between any two material points in it can be considered as infinitesimally short. We ask for the gravitational waves of the system emitted in the direction of the positive x-axis of the system.

[p. 162] The last-named restriction implies that we can replace (7) for a sufficiently large distance R of a point under consideration from the coordinate origin, with the equation

[26]
$$\gamma'_{\mu\nu} = -\frac{\kappa}{2\pi R}\int T_{\mu\nu}(x_\circ, y_\circ, z_\circ, t-R)\,dV_\circ. \qquad (7a)$$

We can limit our consideration to energy-transporting waves; we then only have to form, due to the results of §3, the components γ'_{23} and $\frac{(\gamma'_{22}-\gamma'_{33})}{2}$. The space

[27] integrals on the right-hand side of (7a) can be rewritten in a way which M. Laue has devised. We only want to give a detailed calculation of the integral

$$\int T_{23}\,dV_\circ.$$

[28] Multiplying the two momentum equations

$$\frac{\partial T_{21}}{\partial x_1}+\frac{\partial T_{22}}{\partial x_2}+\frac{\partial T_{23}}{\partial x_3}+\frac{\partial T_{24}}{\partial x_4} = \sigma$$

[29]
$$\frac{\partial T_{31}}{\partial x_1}+\frac{\partial T_{32}}{\partial x_2}+\frac{\partial T_{33}}{\partial x_3}+\frac{\partial T_{34}}{\partial x_4} = \sigma$$

by $\frac{x_3}{2}$ and $\frac{x_2}{2}$, resp., then integrating both over the entire material system and adding them together, one gets, by partial integration after a simple rearrangement,

$$-\int T_{23}\,dV_\circ + \frac{1}{2}\frac{d}{dx_4}\left\{\int (x_3 T_{24} + x_2 T_{34})\,dV_\circ\right\} = 0.$$

We transform the latter integral again with the help of the energy equation

$$\frac{\partial T_{41}}{\partial x_1}+\frac{\partial T_{42}}{\partial x_2}+\frac{\partial T_{43}}{\partial x_3}+\frac{\partial T_{44}}{\partial x_4} = 0.$$

We multiply this equation by $\frac{x_2 x_3}{2}$, again integrate, and rearrange with partial integration. We obtain

$$-\frac{1}{2}\int(x_3 T_{42} + x_2 T_{43})dV_\circ + \frac{1}{2}\frac{d}{dx_4}\left\{\int x_2 x_3 T_{44} dV_\circ\right\} = 0.$$

When this is substituted into the equation above, one gets

$$\int T_{23} dV_\circ = \frac{1}{2}\frac{d^2}{dx_4^2}\left\{\int x_2 x_3 T_{44} dV_\circ\right\},$$

or, since $\frac{d^2}{dx_4^2}$ must be replaced by $-\frac{d^2}{dt^2}$, and T_{44} by the negative density $(-\rho)$ of matter: [p. 163]

$$\int T_{23} dV_\circ = \frac{1}{2}\ddot{\mathfrak{I}}_{23}. \tag{22}$$

We used here the abbreviation

$$\mathfrak{I}_{\mu\nu} = \int x_\mu x_\nu \rho dV_\circ. \tag{23}$$

The $\mathfrak{I}_{\mu\nu}$ are the components of the (time-variable) momentum of inertia of the material system.

In an analogous manner one gets

$$\int(T_{22} - T_{33})dV_\circ = \frac{1}{2}(\ddot{\mathfrak{I}}_{22} - \ddot{\mathfrak{I}}_{33}). \tag{24}$$

On the basis of (22) and (24) one finds from (7a)

$$\gamma'_{23} = -\frac{\kappa}{4\pi R}\ddot{\mathfrak{I}}_{23} \tag{25}$$

$$\frac{\gamma'_{22} - \gamma'_{33}}{2} = -\frac{\kappa}{4\pi R}\left(\frac{\ddot{\Im}_{22} - \ddot{\Im}_{33}}{2}\right). \qquad (26)$$

According to (7a), (22), (24), the $\Im_{\mu\nu}$ are to be taken at time $t - R$, that is, as functions of $t - R$, or, for large R in the vicinity of the x-axis, also as functions of $t - x$. Thus, (25), (26) represent gravitational waves whose energy flux along the x-axis has, according to (16), the density

[30]

$$\frac{t_{41}}{i} = \frac{\kappa}{64\pi^2 R^2}\left[\left(\frac{\dddot{\Im}_{22} - \dddot{\Im}_{33}}{2}\right)^2 + \dddot{\Im}_{23}{}^2\right]. \qquad (27)$$

We also pose the problem to calculate the overall radiation of gravitational waves originating from the system. In order to answer this question, we first ask what the energy radiation of the mechanical system is with respect to a directionally defined cosine α. This can be found by a transformation or, shorter, by reducing it to the following formal problem.

Let $A_{\mu\nu}$ be a symmetric tensor (in three dimensions) and α_ν a vector. Find a scalar S as a function of $A_{\mu\nu}$ and of α_ν in which $A_{\mu\nu}$ occurs integer and homogeneous of second degree, such that S reduces for $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 0$ to

[p. 164] $\left(\frac{A_{22} - A_{33}}{2}\right)^2 + A_{23}^2$. The desired scalar will be a function of the scalars $\sum_\mu A_{\mu\mu}$, $\sum_{\mu\nu} A_{\mu\nu}^2$, $\sum_{\mu\nu} A_{\mu\nu}\alpha_\mu\alpha_\nu$, $\sum_{\mu\sigma\tau} A_{\mu\sigma}A_{\mu\tau}\alpha_\sigma\alpha_\tau$. Since the last two scalars reduce for $\alpha_\nu = (1, 0, 0)$ to A_{11} and $\sum_\mu A_{1\mu}^2$ resp., one finds, after some reflection, that the scalar desired is

$$\begin{aligned} S = &-\frac{1}{4}\left(\sum_\mu A_{\mu\mu}\right)^2 + \frac{1}{2}\sum_\mu A_{\mu\mu}\sum_{\rho\sigma} A_{\rho\sigma}\alpha_\rho\alpha_\sigma \\ &+ \frac{1}{4}\left(\sum_{\rho\sigma} A_{\rho\sigma}\alpha_\rho\alpha_\sigma\right)^2 + \frac{1}{2}\sum_{\mu\nu} A_{\mu\nu}^2 - \sum_{\mu\sigma\tau} A_{\mu\sigma}A_{\mu\tau}\alpha_\sigma\alpha_\tau. \end{aligned} \qquad (28)$$

It is clear that S is the density of the radially flowing gravitational radiation toward the "outside" in the direction $(\alpha_1, \alpha_2, \alpha_3)$, provided one puts

$$A_{\mu\nu} = \frac{\sqrt{\kappa}}{8\pi R}\dddot{\mathfrak{I}}_{\mu\nu}. \qquad (29)$$

If one forms the mean value of S over all directions of space for a fixed value of $A_{\mu\nu}$, one obtains the mean density $\bar{S}$ of the radiation. Finally, $\bar{S}$ multiplied by $4\pi R^2$ is the energy loss (per time unit) of the mechanical system due to gravitational waves. The calculation finds

$$4\pi R^2\bar{S} = \frac{\kappa}{80\pi}\left[\sum_{\mu\nu}\dddot{\mathfrak{I}}^{\,2}_{\mu\nu} - \frac{1}{3}\left(\sum_{\mu}\dddot{\mathfrak{I}}_{\mu\mu}\right)^2\right]. \qquad (30)$$ [31]

This result shows that a mechanical system which permanently retains spherical symmetry cannot radiate; this is in contrast to the result of the previous paper, marred by an error in calculation. [32]

One sees from (27) that the emission cannot turn negative in any direction; consequently, the total emission certainly cannot turn negative, either. It has already been emphasized in a previous paper that the end result of this investigation—which would require a loss of energy of bodies due to the thermal agitation—must raise doubts as to the general validity of the theory. It seems that a more complete quantum theory would also have to bring about a modification of the theory of [33] gravitation.

5. Effect of Gravitational Waves upon Mechanical Systems

For the sake of completeness, we briefly want to consider how far energy from gravitational waves can be absorbed by mechanical systems. Let there again be a mechanical system like the one investigated in §4. Let this system be acted upon by [p. 165] gravitational waves whose wavelength is large compared to the extension of the system. In order to find out about the energy absorption of the system, we look at the energy-momentum equation of matter,

$$\sum_{\sigma}\frac{\partial \mathfrak{T}_{\mu}^{\sigma}}{\partial x_{\sigma}}+\frac{1}{2}\sum_{\rho\sigma}\frac{\partial g^{\rho\sigma}}{\partial x_{\mu}}\mathfrak{T}_{\rho\sigma} = 0 .$$

We integrate this equation at constant x_4 over the entire system and obtain for $\mu = 4$ (energy theorem)

$$\frac{d}{dx_4}\left\{\int \mathfrak{T}_4^4 \, dV\right\} = -\frac{1}{2}\int dV \sum_{\rho\sigma}\frac{\partial g^{\rho\sigma}}{\partial x_4}\mathfrak{T}_{\rho\sigma} .$$

The integral on the left-hand side is the energy E of the whole material system. On the left, therefore, we have the energy increase in time. If the differentiation is done with respect to real-valued time, and if one limits oneself on the right-hand side to keeping terms of second order of magnitude, one gets

$$\frac{dE}{dt} = \frac{1}{2}\int dV \sum_{\rho\sigma}\left(\frac{\partial \gamma_{\rho\sigma}}{\partial t}T_{\rho\sigma}\right) . \tag{31}$$

Now we can split the $\gamma_{\rho\sigma}$, which represent the gravitational field, into a part $(\gamma_{\rho\sigma})_w$ corresponding to the incident wave, and another constituent $(\gamma_{\rho\sigma})_v$ according to the equation

$$\gamma_{\rho\sigma} = (\gamma_{\rho\sigma})_w + (\gamma_{\rho\sigma})_v . \tag{32}$$

Accordingly, the integral on the right-hand side of (31) splits into a sum of two integrals of which the first one represents the energy increase originating from the wave. This is the only part of interest here; thus, in order not to be burdened by complicated notation, we shall henceforth interpret (31) such that $\frac{dE}{dt}$ represents only the energy increase from the wave, and for the portion $(\gamma_{\rho\sigma})_w$ we merely write $\gamma_{\rho\sigma}$. This $\gamma_{\rho\sigma}$ is now a locally slowly variable function and we are entitled to write

$$\frac{dE}{dt} = \frac{1}{2}\sum_{\rho\sigma}\frac{\partial \gamma_{\rho\sigma}}{\partial t} \cdot \int T_{\rho\sigma} dV . \tag{33}$$

Let the acting wave be an energy-transporting one in which only the component $\gamma_{23}(=\gamma'_{23})$ of the gravitational field is different form zero. Because of (22) one then has

$$\frac{dE}{dt} = \frac{1}{2}\frac{\partial\gamma_{23}}{\partial t}\frac{d^2\mathfrak{J}_{23}}{dt^2}. \qquad (34)$$ [34]

For a given wave and given mechanical process, the energy absorbed from the wave [p. 166]
can, therefore, be found by integration.

§6. Answer to an Objection of Mr. Levi-Civita

In a series of interesting investigations, Mr. Levi-Civita has in recent times contributed to clarification of problems in the general theory of relativity. In one of these papers[6] he takes, with respect to the conservation theorems, a position which differs from my point of view and denies, based upon his interpretation, the claim of my conclusions regarding the radiation of energy through gravitational waves. Even though we have in the meantime, through an exchange of letters, clarified the question in a manner satisfactory to both of us, I consider it advisable in the interest of the subject matter to add a few general remarks about the conservation theorems. [35] [37]

It is generally conceded that, according to the foundations of the general theory of relativity, there exists a four-equation, valid under an arbitrary choice of the system of reference, and it has the form

$$\sum_{\nu}\frac{\partial(\mathfrak{T}_{\sigma}^{\nu} + t_{\sigma}^{\nu})}{\partial x_{\nu}} = 0 \quad (\sigma = 1,2,3,4), \qquad (35)$$

where $\mathfrak{T}_{\sigma}^{\nu}$ are the energy components of matter and the t_{σ}^{ν} are functions of the $g_{\mu\nu}$ and their *first* derivatives. But there are differences of opinion on whether or not the t_{σ}^{ν} have to be interpreted as the energy components of the gravitational field. I consider this difference of opinions as irrelevant, as a mere question of words. However, I claim that the undisputed equation, given above, provides the facilitations of perspective which constitutes the value of the conservation

[6] *Accademia dei Lincei* 26 (April 1, 1917). [36]

theorems. I want to explain this with the fourth equation ($\sigma = 4$), which I am used to calling the energy equation.

Let there be given a spatially limited material system outside of which densities of matter and electromagnetic field strengths vanish. We imagine a surface S at rest, embracing this entire material system. Integrating the fourth equation over the entire space contained by S, one obtains:

[38][6]

$$-\frac{d}{dx_4}\left\{\int(\mathfrak{T}_4^4 + t_4^4)dV\right\}$$

$$= \int_S (t_4^1 \cos(nx_1) + t_4^2 \cos(nx_2) + t_4^3 \cos(nx_3))d\sigma. \qquad (36)$$

Nobody can be compelled, by any reason whatsoever, to call t_4^4 the energy density of the gravitational field and (t_4^1, t_4^2, t_4^3) the components of the energy flux of
[p. 167] gravitation. But one can claim the following: the right-hand side definitely represents the loss of material energy in the system if the space integral of t_4^4 is small
[39] compared to the one of the "material" energy density $\mathfrak{T}_4^4$. It is this point alone which I have used in my present and in past papers on gravitational waves.

Mr. Levi-Civita (and before him, with less emphasis, already H. A. Lorentz) has suggested a formulation of the conservation theorems that deviates from (35). He
[40] (and with him other colleagues) is opposed to an emphasis of equations (35), and are also opposed to the above interpretation because the t_σ^ν do not form a *tensor*. The latter is readily conceded; but I do not understand why only quantities with the transformation characteristics of tensor components should be granted physical meaning. Necessary is only that equation systems are valid for any choice of a system of reference which for the equation system (35) is true. Levi-Civita suggests the following formulation for the energy-momentum theorem. He writes the field equations of gravitation in the form

[41]

$$T_{im} + A_{im} = 0, \qquad (37)$$

where T_{im} is the energy tensor of matter and A_{im} is a covariant tensor that depends only upon the $g_{\mu\nu}$ and their first two derivatives with respect to the coordinates. The A_{im} are called the energy components of the gravitational field.

A *logical* objection can, of course, not be raised against such wording. But I find that (37) does not allow us to draw these conclusions which we are used to drawing from the conservation theorems. This is connected to the fact that in (37) the components of the *total energy* vanish everywhere. The equations (37), for example, do not exclude the possibility (and this in contrast to the equations [35]) that a material system dissolves into just nothing without leaving any trace. Because the total energy in (37)—but not in (35)—is zero from the beginning: the conservation of this energy value does not demand the continued existence of the system in any form.

Translator's Notes

{1} The index σ was omitted.

{2} In front of the last derivative of γ', a corrective $\sum_{\lambda}$, has been inserted.

{3} $\delta_{\rho\sigma}$ has been corrected to $\delta_{\mu\sigma}$.

{4} $\gamma^{1}_{\mu\nu}$ has been corrected to $\gamma'_{\mu\nu}$.

{5} The $i\lambda_2$ of the first term in the fourth row has been corrected to $i\lambda_1$.

{6} $\overset{3}{t}_{43}$ in the last cos-term has been corrected to $\overset{3}{t}_{4}$.